Proof-theoretic approach to representable qualitative probabilities

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1 Introduction

Logical systems for uncertain reasoning, based on *comparisons* of plausibility of events are widespread in the literature [9]. These systems have often been introduced as a more realistic counterpart to standard probabilistic models, which seem to require that agents are capable of assigning precise real values in [0,1] to events to quantify their uncertainty.

Particular comparative models, known as *qualitative probabilities* have been presented already in the early literature [7, 5], as a foundation for, rather than an alternative to, standard probabilities. In this respect, researchers are interested in providing axiom systems that use a binary connective for comparisons of events and are complete over a semantics based on comparisons of standard probabilistic functions. This is a logical reframing of the problem of representation in the classical literature on measurement theory [13].

Since [11], it has been well known that the standard axioms for qualitative probability introduced in [5] do not suffice for probabilistic completeness. Different axiomatic systems have since been provided, see [10] a recent survey. Each such system is presented in axiomatic form and include a non standard axiom: for instance one may consider a schema of infinitely many axioms (the so-called *cancellation* axioms [12]), or the so-called *polarization rule* [2], which require the use of additional variables in the language.

Computationally, despite the apparent simplicity of qualitative reasoning, systems for reasoning with qualitative probability are no less complex (satisfiabiliy is NP-complete) than systems for standard probabilistic reasoning based on linear inequalities [10, 9].

In this work we introduce proof-theoretic tools to the investigation of logics of qualitative probability which, to the best of our knowledge are missing in the literature. We present a tableaux-style calculus, which is complete w.r.t the probabilistic semantics and report on ongoing work, introducing weaker systems for qualitative probability that have *polynomial* computational complexity and approximate full qualitative reasoning, based on the model of the polynomial approximation of classical logic in [4].

2 The framework

2.1 Language

We adopt a two-layered language [8, 6, 1] designed for reasoning about uncertainties, separating reasoning about events (lower layer) from reasoning about uncertainties of these events. Both layers are connected by an operator expressing uncertainty. The system is modular in the sense that each layer can use a different logic. We will use classical logic on both layers with an operator for comparison of probabilities. Formulas of the lower layer for events are built by classical propositional variables $p, q, \ldots$ closing off under the usual classical connectives and constants, i.e. according to the following BNF grammar

$$\varphi ::= \bot \mid \top \mid p \mid \varphi \wedge \varphi \mid \neg\varphi$$

We denote such classical formulas by Fm_{CL}. On top of classical formulas, we then consider a layer of *comparison* formulas Fm_{Comp}, obtained via the following BNF grammar

$$\alpha ::= \varphi \preceq \varphi \mid \alpha \wedge \alpha \mid \neg\alpha$$

where φ is a classical formula in Fm_{CL}. We adopt the convention of abbreviating by $\varphi \prec \psi$ the formula $\neg(\psi \preceq \varphi)$ and by $\varphi \approx \psi$ the formula $(\varphi \preceq \psi) \wedge (\psi \preceq \varphi)$.

2.2 Semantics

The semantics is given in our setting by a probability function over classical formulas.

Definition 1. *Let* $\models_{CL}$ *the consequence relation of classical logic. A probability function is a function* $\mathrm{p}\colon Fm_{CL} \to [0,1]$ *such that:*

- *if* $\models_{CL} \alpha$ *then* $\mathrm{p}(\alpha) = 1$
- *if* $\alpha \models_{CL} \beta$ *then* $\mathrm{p}(\alpha) \leq \mathrm{p}(\beta)$.
- *if* $\alpha \wedge \beta \models_{CL} \bot$ *then* $\mathrm{p}(\alpha \vee \beta) = \mathrm{p}(\alpha) + \mathrm{p}(\beta)$

We are now ready to define the semantic notion for the whole language Fm_{Comp}

Definition 2. *(Probabilistic model) Let* p *be a probability function on* Fm_{CL}. M_p *is a probabilistic model based on* p *where the satisfaction relation is defined recursively as follows* ($\varphi, \psi \in Fm_{CL}, \alpha, \beta \in Fm_{Comp}$)*:*

- $M_\mathrm{p} \models \varphi \preceq \psi$ *iff* $\mathrm{p}(\varphi) \leq \mathrm{p}(\psi)$.
- $M_\mathrm{p} \models \alpha \wedge \beta$ *iff* $M_\mathrm{p} \models \alpha$ *and* $M_\mathrm{p} \models \beta$
- $M_\mathrm{p} \models \neg\alpha$ *iff* $M_\mathrm{p} \not\models \alpha$

$$\frac{\varphi \land \psi}{\begin{array}{c}\varphi \\ \psi\end{array}} \qquad \frac{\neg(\varphi \land \psi) \qquad \varphi}{\neg\psi}$$

$$\frac{\varphi \lor \psi \qquad \neg\varphi}{\psi} \qquad \frac{\neg(\varphi \lor \psi)}{\begin{array}{c}\neg\varphi \\ \neg\psi\end{array}}$$

$$\frac{}{\varphi \mid \neg\varphi} \qquad \frac{\begin{array}{c}\varphi \\ \neg\varphi\end{array}}{\times} \qquad \frac{\neg\neg\varphi}{\varphi}$$

$$\frac{\bot}{\times} \qquad \frac{}{\top}$$

Table 1: The rules of a KE-style calculus for classical logic

2.3 Proof systems for qualitative reasoning

We now present our tableaux-style system for qualitative reasoning. Our approach is based on so-called KE calculus for classical logic, which is an efficient proof-system [3] containing only one branching rule that intuitively implements a Principle of Bivalence, and essentially corresponds to a non-eliminable cut rule. The rest of the rules have all a non-branching format and correspond to traditional reasoning patterns. The system is recalled in Table 1.

Our KE-style calculus KE_{quasi} for qualitative probability is then obtained in the following way:

- The rules in Table 2
- The rules of KE_{CL} in Table1, applicable to formulas in the upper layer and in the lower layer, within a box.

As it is usual for Tableaux, a proof of a formula γ in KE_{quasi} is just a closed tree rooted in $\neg\gamma$.

The calculus KE_{quasi} is shown in our paper to be equivalent to the axiomatic system AX_{quasi} [10], which consists of a complete axiom system for classical logic plus

(i) if $\varphi \models \psi$ then $\varphi \preceq \psi$

(ii) $((\varphi \preceq \psi) \land (\psi \preceq \chi)) \to (\varphi \preceq \chi)$

(iii) $(\varphi \preceq \psi) \lor (\psi \preceq \varphi)$

(iv) $\bot \prec \top$

(v) $(\varphi \preceq \psi) \leftrightarrow (\varphi \wedge \neg\psi \preceq \psi \wedge \neg\varphi)$

Note that the boxed rule $contr_{CL}$ in Table 2 corresponds to axiom (i), connecting the classical consquence relation and qualitative probability. The idea is that derivation inside the box is to be performed within KE_{CL}. If the derivation is closed, one can "discharge" such derivation and obtain that the formula $\varphi \prec \psi$ delivers a contradiction. Note that, as for KE_{CL}, derivation in KE_{quasi} have only one branching rule, (PB), corresponding to classical bivalence.

Both KE_{quasi} and AX_{quasi} are however, sound but not complete, with respect to the standard probabilistic semantics. In order to achieve completeness, we need additional rules, corresponding to the so-called *polarization* axiom. In terms of Hilbert-style calculi, polarization can be presented as:

$$\frac{(\varphi \wedge a \approx \varphi \wedge \neg a) \rightarrow \gamma}{\gamma}$$

The crucial point here is that a is a propositional variable not occurring in φ, γ, and essentially $\varphi \wedge a$ and $\varphi \wedge \neg a$ amount to an event "dividing in half" φ in terms of probability.

We follow a different approach to polarization here, enriching first our language, by introducing a unary connective h, such that $h\varphi$ is interpreted as "a subevent of φ having probability half of φ". Denoting by $h^*\varphi$ the formula $\varphi \wedge \neg h\varphi$, we then obtain that for any probability function $\mathrm{p}(h\varphi) = \mathrm{p}(h^*\varphi) = \frac{1}{2}\mathrm{p}(\varphi)$.

This motivates the introduction of our calculus KE_{qual}, which is composed of:

- The calculus KE_{quasi}
- the rule to be added to the rules of KE_{CL} for the lower layer (i.e. the boxed formulas):

$$\frac{\begin{array}{c} h\varphi \\ \neg\varphi \end{array}}{\times}\ (contr_h).$$

- the rules in the upper layer

$$\frac{h\varphi \prec h^*\varphi}{\times}\ (h_1)$$

$$\frac{h^*\varphi \prec h\varphi}{\times}\ (h_2).$$

Our main result shows that KE_{quasi} is sound and complete w.r.t. the probabilistic semantics. Our polynomial approximation is then obtained by suitable restriction of the rule (PB) and the branching rule in Table 1 and 2, respectively.

$$(taut)\ \frac{}{\bot \prec \top} \qquad (tr)\ \frac{\varphi \preceq \psi \quad \psi \preceq \chi}{\varphi \preceq \chi} \qquad (PB)\ \frac{}{\varphi \preceq \psi \quad | \quad \psi \prec \varphi}$$

$$(add_{\prec})\ \frac{\varphi \wedge \neg\psi \prec \psi \wedge \neg\varphi}{\varphi \prec \psi} \qquad (add_{\preceq})\ \frac{\varphi \wedge \neg\psi \preceq \psi \wedge \neg\varphi}{\varphi \preceq \psi}$$

$$(contr_{CL})\ \frac{\begin{array}{c}\varphi \prec \psi \\ \boxed{\begin{array}{c}\psi \\ \neg\varphi \\ \vdots \\ \times\end{array}}\end{array}}{\times} \qquad (contr_{\preceq})\ \frac{\varphi \preceq \psi \quad \psi \prec \varphi}{\times} \qquad (contr_{\prec})\ \frac{\varphi \prec \psi \quad \psi \prec \varphi}{\times}$$

Table 2: The rules of KE$_{quasi}$ for qualitative reasoning

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